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BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL FORMS ON COMPACT RIEMANNIAN MANIFOLDS

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Abstract: In this paper, solutions of boundary value problems characterizing harmonic fields defined on Riemannian manifolds with boundary and the corresponding inhomogeneous generalizations will be presented. These problems are related with the static Maxwell equations if vector fields on three dimensional manifolds are considered. For our potential theoretical approach, we will prove a generalized fundamental theorem for differential forms on submanifolds with boundary. As it is well-known, the corresponding theorem for vector valued functions in Euclidean spaces is extensively used for boundary value problems in Electrodynamics. Dirichlet and Neumann boundary value problems in $\mathbf{R}^n$ can be solved by means of Fredholm integral equations which are derived from the fundamental theorem. Furthermore, harmonic fields play an important role with respect to topological considerations.

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1 Introduction

This article offers a new fundamental theorem for differential forms on Riemannian submanifolds with boundary. Our theorem represents a generalization of results shown in [8, 9]. Kress gives there a fundamental theorem for vector fields on $\mathbf{R}^3$ and skew-symmetric tensor fields on $\mathbf{R}^n$. We will prove that r-forms f, defined on submanifolds Ω and of regularity class $C^{1,\lambda}(\bar{\Omega})$, may be decomposed into

$$f - Hf = d\phi + \delta\theta\,,$$

where

$$\phi(x) := -\int_{\partial\Omega} G_{r-1,r-1}(x,y) \wedge *_y f(y) + \int_{\Omega} G_{r-1,r-1}(x,y) \wedge *_y \delta_y f(y)$$

$$\theta(x) := -\int_{\partial\Omega} f(y) \wedge *_y G_{r+1,r+1}(x,y) + \int_{\Omega} d_y f(y) \wedge *_y G_{r+1,r+1}(x,y)\,.$$

Here d is the exterior derivative, δ is the coderivative, and $*$ denotes the Hodge mapping. $G_{r,r}$ is a given double form and Hf is a harmonic field. Among other things, these decomposition results provide solutions of important boundary value problems and the accompanying a priori estimates, as shown in [8, 9, 12] and [3, 4, 5, 13]. We find in [8, 9, 12] recipes to derive solutions of homogeneous and inhomogeneous Dirichlet and Neumann boundary value problems for vector fields using the fundamental theorem in $\mathbb{R}^n$. In [9] this concept is generalized to skew-symmetric tensor fields.

At first glance, the fundamental theorem presented here seems to be a particular case of the Hodge-Kodaira-Morrey decomposition (cf. [10]). But this decomposition for differential forms defined on manifolds with or without boundary will not provide a direct approach to the usual boundary problems of Electrostatics and Magnetostatics. We have to be aware of major differences between both decompositions. This can be illustrated for vector fields in $\Omega \subset \mathbb{R}^3$ for example. In this case, the Hodge-Kodaira-Morrey decomposition is given by a refined Helmholtz decomposition. The latter one is the direct sum

$$L^p(\Omega, \mathbb{R}^3) = \{\nabla f \mid f \in W^{1,p}(\Omega)\} \oplus \overline{\{g \in C_0^\infty(\Omega, \mathbb{R}^3) \mid div\, g = 0\}}^{\|\cdot\|_p}.$$

As shown in [12], $L^p(\Omega, \mathbb{R}^3)$ can be expressed by the ranges $\mathcal{R}$ of projection operators Q and P, i.e.

$$L^p(\Omega, \mathbb{R}^3) = \mathcal{R}(Q) + \mathcal{R}(P).$$

If $f \in C_0^\infty(\Omega, \mathbb{R}^3)$, we are able to explicitly define these operators by

$$(Qf)(x) := -\frac{1}{4\pi}\, grad \int\limits_\Omega \frac{1}{|x-y|}\, div_y\, f(y)\, dy + grad\, H(x)$$

and

$$(Pf)(x) := \frac{1}{4\pi}\, curl \int\limits_\Omega \frac{1}{|x-y|}\, curl_y\, f(y)\, dy - grad\, H(x),$$

where H is a solution of a particular Neumann problem. The fundamental theorem for vector fields $g \in C^1(\Omega, \mathbb{R}^3) \cap C^0(\bar{\Omega}, \mathbb{R}^3)$ provides the representation

$$g = -grad\, U + curl\, A,$$

where U is given by the scalar potential

$$U(x) := \frac{1}{4\pi} \int\limits_\Omega \frac{1}{|x-y|}\, div_y\, g(y)\, dy - \int\limits_{\partial\Omega} \frac{(\nu, g)(y)}{|x-y|}\, d\omega_y$$

and A by the solenoidal vector potential

$$A(x) := \frac{1}{4\pi} \int\limits_\Omega \frac{1}{|x-y|}\, curl_y\, g(y)\, dy - \int\limits_{\partial\Omega} \frac{[\nu, g](y)}{|x-y|}\, d\omega_y.$$

For vector fields $f \in C_0^\infty(\Omega, \mathbf{R}^3)$ this implies

$$f(x) = -\frac{1}{4\pi}\, grad \int\limits_\Omega \frac{1}{|x-y|}\, div_y\, f(y)\, dy + \frac{1}{4\pi}\, curl \int\limits_\Omega \frac{1}{|x-y|}\, curl_y\, f(y)\, dy\,.$$

Hence this decomposition generally does not use projection operators in order to define the relevant subspaces. The representation of the fundamental theorem differs from the Helmholtz decomposition by an artifical term $grad\, H$.

We will generalize results shown in [9, 12] in order to find solutions of boundary value problems for differential forms defined on Riemannian manifolds with boundary. The corresponding boundary value problems include problems which characterize harmonic fields, i.e. define Neumann fields if the normal components will vanish and define Dirichlet fields if the tangential components will vanish.

By means of the fundamental theorem it can be shown that the r-form

$$d \int\limits_{\partial\Omega} (G_{r-1,r-1}(x,y), \epsilon(y))_y\, d\omega_y$$

is a Dirichlet field in Ω if ϵ is a solution of a homogeneous Fredholm equation which is related to the Fredholm equations of the scalar Dirichlet and Neumann Laplace problem.

Our afore mentioned boundary value problems for harmonic fields can be generalized to non-harmonic field equations as given in [9] for skew-symmetric r-fields on $\mathbf{R}^n$. Similar problems on Riemannian manifolds with boundary are presented in [6]. The solution method there might be considered to be less direct than the following one. Both methods distinctly differ from each other.

2 Definitions and Preliminaries

DEFINITION 1

a) *For $n \in \mathbb{N} : n \geq 2$, let $\mathcal{M} = \mathcal{M}^n$ be a compact, oriented n dimensional C^∞-Riemannian manifold, let $\Omega = \Omega^n \subset\subset \mathcal{M}$ be an oriented n dimensional C^∞-Riemannian submanifold with boundary which consists of a finite number of arcwise connected domains with C^∞ boundaries and pairwise disjunct closures.*

b) *Let g be the Riemannian metric, g_{ij} its covariant components and g^{ij} its contravariant components. If f, h are r-forms, where $0 < r \leq n$, then the inner product (f,h) is defined by*

$$(f,h) = (h,f) := \frac{1}{r!} \sum_{j_1 \ldots j_r, l_1 \ldots l_r} g^{j_1 l_1} \ldots g^{j_r l_r} f_{j_1 \ldots j_r} h_{l_1 \ldots l_r}\,.$$

For 0-forms f and h, we define $(f,h) = (h,f) := f \cdot h$.

Let now f be a 1-form, and h be an r-form. For $0 < r \leq n$, the $(r-1)$-form (f,h) is given in local coordinates by

$$(f,h) = (h,f) := \frac{1}{r!} \sum_{l_2 \ldots l_r} (f,h)_{l_2 \ldots l_r} dx^{l_2} \wedge \ldots \wedge dx^{l_r}, \text{ where}$$

$$(f,h)_{l_2 \ldots l_r} := \sum_{k,l=1}^{n} f_k g^{kl} h_{l l_2 \ldots l_r}.$$

If h is a 0-form, then it is convenient to set for this form

$$(f,h) = (h,f) := 0.$$

c) *Let f be a differential form, let $\rho = \rho(x,y)$ the geodesic distance and let $k \in \mathbb{R}$. Then $f(x,y) = O(\rho^k)$ expresses concisely that $\rho^{-k} \cdot f(x,y)$ is bounded with regard to each component.*

For a linear space L, the symbol L^r denotes the space of r-forms with components in L. Let $(V_k, \varphi_k, W_k)_{k \in K}$ be the atlas of an n dimensional Riemannian manifold M with or without boundary, and let $(\psi_k)_{k \in K}$ be the corresponding partition of unity. Then we will write $f \in L(M)$ if $\sum_{k \in K} \psi_k f \circ \varphi_k^{-1} \in L(\mathbb{R}^n)$.

For spaces $C^{k,\lambda}$ we presume that $k \in \mathbb{N}_0$ and $0 < \lambda < 1$. The expressions $C^{k,\lambda}(\partial\Omega)^r$ and $C^\infty(\partial\Omega)^r$ are used as abbreviations for $C^{k,\lambda}(\bar{\Omega})^r|_{\partial\Omega}$ and $C^\infty(\bar{\Omega})^r|_{\partial\Omega}$, respectively.

The 1-form $\nu = \nu(x)$ denotes the differential form given by components which equal the components of the exterior normal in $x \in \partial\Omega$. For a differential form f defined on $\partial\Omega$, we call (ν, f) the normal component of f and $\nu \wedge f$ the tangential component of f. Subscripts ν or τ of $C^{k,\lambda}(\partial\Omega)^r$ and $C^\infty(\partial\Omega)^r$ mean that $(\nu, f) = 0$ or $\nu \wedge f = 0$ for forms f of the space in question.

The Lebesgue L^p-spaces for r-forms defined on $\partial\Omega$ will be understood as completion of smooth forms on $\bar{\Omega}$ with regard to $\|.\|_p$. Correspondingly $L^p_\nu(\partial\Omega)^r$ and $L^p_\tau(\partial\Omega)^r$ are completions of the respective smooth subspaces $C^\infty_\nu(\partial\Omega)^r$ and $C^\infty_\tau(\partial\Omega)^r$.

Let $0 \leq r < n$, $f \in L^2_\nu(\partial\Omega)^r$ and $g \in L^2_\tau(\partial\Omega)^{r+1}$. Then the bilinear form $\langle f, g \rangle$ is defined by

$$\langle f, g \rangle := \int_{\partial\Omega} (f(x), (\nu(x), g(x)))\, d\omega_x.$$

According to common notation, d designates the exterior derivative, and δ the coderivative. For $f \in C^1(M)^r$ we set

$$\delta f := (-1)^{(r+1)n} * (d(*f)) \quad \text{if } r > 0,$$

and $\delta f = 0$ *if* $r = 0$.

By $*$ *the Hodge mapping is denoted.*

Harmonic fields $f \in C^{1,\lambda}(\bar{\Omega})^r$, *fulfilling* $(\nu, f) = 0$, *will be called Neumann fields* $\mathcal{Z}(\Omega)^r$ *on* Ω.

Correspondingly, harmonic fields $f \in C^{1,\lambda}(\bar{\Omega})^r$, *obeying* $\nu \wedge f = 0$, *are denoted as Dirichlet fields* $\mathcal{Y}(\Omega)^r$ *on* Ω.

3 The Fundamental Theorem for Differential Forms on Compact Riemannian Manifolds

The fundamental theorem for vector fields (cf. [8, 12]) is extensively used to solve basic boundary value problems given in Electrostatics, Magnetostatics as well as Hydrostatics. In [9], Kress presents a generalization of this fundamental theorem for $\mathbf{R}^n$-fields and investigates solutions of suitable generalized boundary value problems. The following chapters again offer a generalization, but now with regard to the presupposed manifold.

According to [11], we define:

DEFINITION 2: *Let* $\mathcal{M}$ *be an oriented* n *dimensional* C^∞*-Riemannian manifold. The mapping* $j(x,y) = j(y,x) \in C^\infty(\mathcal{M} \times \mathcal{M})$ *is presumed to satisfy* $0 \leq j(x,y) \leq 1$. *This function* $j \in C_0^\infty(W)$ *is equal to 1 in a neighbourhood* $W_0 \subset\subset W$ *of the diagonal, where* W *is defined according to [11, p. 114]. The constant* ω_n *designates the* $(n-1)$ *dimensional surface area of the unit sphere in* $\mathbf{R}^n$, *and* $\alpha(x,y)$ *denotes the double form*

$$\alpha(x,y) = \alpha_{r,r}(x,y) := \frac{1}{r!}(d_x d_y(-\frac{1}{2}\rho^2(x,y)))^r .$$

By means of this double form, the form $\omega(x,y) = \omega_{r,r}(x,y)$ *is defined by*

$$\omega_{r,r}(x,y) := \begin{cases} -\dfrac{1}{(n-2)\omega_n \rho^{n-2}(x,y)} \cdot j(x,y) \cdot \alpha(x,y) & \text{for } n > 2 \\[2ex] \dfrac{1}{2\pi} \ln \rho(x,y) \cdot j(x,y) \cdot \alpha(x,y) & \text{for } n = 2 . \end{cases}$$

When dealing with the Euclidean manifold $\mathbf{R}^n$, it is interesting to understand how the results of Kress, presented in [8, 9], are embedded in the results of our approach. A number of formulas will be summarized in the following *Remarks 1* and *2* for this purpose.

REMARK 1: *On compact subsets $U \subset \mathbf{R}^n$, the double form $\omega_{r,r}$ might then be written as*

$$\omega_{r,r|U} = (G_0 \sum_{i_1 < \ldots < i_r} dx^{i_1} \wedge \ldots \wedge dx^{i_r} \otimes dy^{i_1} \wedge \ldots \wedge dy^{i_r})_{|U} ,$$

where G_0 denotes the Newton potential function, i.e.

$$G_0(x,y) = \begin{cases} -\dfrac{1}{(n-2)\omega_n |x-y|^{n-2}} & \text{for } n > 2 \\[2ex] \dfrac{1}{2\pi} \ln |x-y| & \text{for } n = 2 . \end{cases}$$

For our approach, it is convenient to define some quantities constituted by the pull back mapping $(\varphi_x^{-1})^*$. We will set $\varphi_x := \exp_x$, where $\exp_x$ denotes the exponential map. Therefore we can benefit from properties of $\exp_x$, concerning geodesic distances and radial isometry within a neighbourhood U_x of $x \in \mathcal{M}$. The range of $\exp_x$ is a subset of the tangential space $T_x\mathcal{M}$, and $T_x\mathcal{M}$ may be equated to a subset of $\mathbf{R}^n$.

The *fundamental theorem* of this paper will provide a representation for differential forms on compact Riemannian manifolds, which generalizes the fundamental theorem of [8, 9]. The following result gives a first hint to the proof of our fundamental theorem. As there is a finite number of linearly independent harmonic forms on a compact Riemannian manifold, the equation

$$\Delta \mu = \beta$$

is not solvable for arbitrary data $\beta \in C^\infty$. This is a major difference compared with the general solvability on the manifold $\mathbf{R}^n$.

LEMMA 1: *Let $0 < r < n$.*

a) There exist operators G and H, constituted by kernels $G(x,y)$ and $H(x,y)$, satisfying

$$(1) \qquad \Delta G\phi = \phi - H\phi \ \text{ for each } \phi \in C^\infty(\mathcal{M})^r .$$

Moreover, the range of G is orthogonal to the harmonic forms, and the double forms $G(x,y)_{|x \neq y}$ and $H(x,y)$ are elements of C^∞.

b) When $1 < p < \infty$, $k \in \mathbf{N}_0$ and $\phi \in L^p(\mathcal{M})^r$, then appropriate extensions of these operators exist and equation (1) is valid for the extensions G and H as well. Here $G\phi$ is part of $W^{2,p}(\mathcal{M})^r$ and $H\phi$ consists of those elements of $L^p(\mathcal{M})^r$ which are harmonic r-forms in the sense of distributions. Furthermore,

$$G \in \mathcal{L}(W^{k,p}(\mathcal{M})^r, W^{k+2,p}(\mathcal{M})^r) \text{ and } H \in \mathcal{L}(W^{k,p}(\mathcal{M})^r, W^{k,p}(\mathcal{M})^r).$$

c) If $k \in \mathbf{N}_0$ and $\phi \in C^{k,\lambda}(\mathcal{M})^r$, where $0 < \lambda < 1$, then

$$G \in \mathcal{L}(C^{k,\lambda}(\mathcal{M})^r, C^{k+2,\lambda}(\mathcal{M})^r) \text{ and } H \in \mathcal{L}(C^{k,\lambda}(\mathcal{M})^r, C^{k,\lambda}(\mathcal{M})^r).$$

Proof: Part a) of the assertion is proven in [11]. For the proof of b) and c), a transformation by $\exp_x$ is helpful. Then one can refer to the well-known estimates for the Newtonian potential and for estimates of weakly singular kernels. In detail, we use the representation

$$G(x,y) = \omega(x,y) + W(x,y)\,, \tag{2}$$

where ω is given in *Definition 2*. The kernel $W(x,y)$ of this representation is continous for $x \neq y$. For $2 \leq n < 4$ this kernel is even everywhere continous. Moreover, there exist a subset U containing x and y, such that $W(x,y)_{|U}$ possesses singularities of type

$$W(x,y)_{|U} = \begin{cases} O(r^{4-n}(x,y)) & \text{if } n > 4 \\ O(r^{-\epsilon}(x,y)),\ \text{for all } \epsilon > 0 & \text{if } n = 4. \end{cases} \tag{3}$$

By transformation of the differential forms $\omega(x,y)$ and $W(x,y)$, we obtain components $\pm\tilde{\omega}_0(p,q)$ and $\pm\tilde{W}_0(p,q)$. Such kernels are extensively treated in [2] with regard to L^p-estimates. As the according $C^{k,\lambda}$-estimates for this type of integral operators are also well-known, the assertions b) and c) are proven.

□

Corresponding results are valid for 0-forms and for n-forms. The relevant kernels $G_{0,0}$ and $H_{0,0}$ are defined in [1].

The range Hf for forms $f \in L^2(\mathcal{M})^r$, $0 \leq r \leq n$, is a subset of $W^{1,2}(\mathcal{M})^r$. This can be shown by formulas presented in [11, chapter 26] for example.

For some of the subsequent considerations, we will extend forms f which are originally defined on sets $\Omega \subset\subset \mathcal{M}$ to $\tilde{f} \in L^2(\mathcal{M})^r$, where $\tilde{f}_{|\Omega} = f$ and $\tilde{f}_{|\hat{\Omega}} = 0$.

Our first main task now is to prove the fundamental theorem presented in

THEOREM 1: Fundamental Theorem

Let $f \in C^1(\bar{\Omega})^r$, where $0 < r < n$. Then the decomposition

$$f - (H\tilde{f})_{|\Omega} = d\phi + \delta\theta\,, \tag{4}$$

constituted by derivations of the differential forms

$$\phi(x) := -\int\limits_{\partial\Omega} G_{r-1,r-1}(x,y) \wedge *_y f_r(y) + \int\limits_{\Omega} G_{r-1,r-1}(x,y) \wedge *_y \delta_y f_r(y)$$

and

$$\theta(x) := -\int\limits_{\partial\Omega} f_r(y) \wedge *_y G_{r+1,r+1}(x,y) + \int\limits_{\Omega} d_y f_r(y) \wedge *_y G_{r+1,r+1}(x,y)\,,$$

is valid.

Moreover, the latter forms have the properties:

$$\delta\phi = 0 \text{ and } d\theta = 0\,,$$

i.e. ϕ *is coclosed and* θ *is closed.*

The form $\tilde{f}$ *is the above mentioned extension.*

Proof: We can conclude from *Lemma 1* that

$$d\delta(G_r f) + \delta d(G_r f) = \triangle(G_r f) = ((1-H)\tilde{f})_{|\Omega} \tag{5}$$

for $f \in C^1(\bar{\Omega})^r$. The equations connecting double forms of different orders given by

$$-\delta_x G_{r,r}(x,y) = d_y G_{r-1,r-1}(x,y) \tag{6}$$

and

$$d_x G_{r-1,r-1}(x,y) = -\delta_y G_{r,r}(x,y) \tag{7}$$

are taken from [11, p. 134]. By means of the formula

$$dg \wedge *h = d(g \wedge *h) - g \wedge *\delta h \tag{8}$$

for forms $g = g_{p-1} \in C^1(\bar{\Omega})^{p-1}$ and $h = h_p \in C^1(\bar{\Omega})^p$, the following equations are derived:

$$\begin{aligned} &d_y G_{r-1,r-1}(x,y) \wedge *_y f_r(y) = \\ &= d_y(G_{r-1,r-1}(x,y) \wedge *_y f_r(y)) - G_{r-1,r-1}(x,y) \wedge *_y \delta_y f_r(y)\,, \end{aligned} \tag{9}$$

where $g_{r-1} = G_{r-1,r-1}$, with x fixed, and $h_r = f_r$, as well as

$$\begin{aligned} &\delta_y G_{r+1,r+1}(x,y) \wedge *_y f_r(y) = \\ &= d_y(f_r(y) \wedge *_y G_{r+1,r+1}(x,y)) - d_y f_r(y) \wedge *_y G_{r+1,r+1}(x,y)\,, \end{aligned} \tag{10}$$

where $h_{r+1} = G_{r+1,r+1}$, with x fixed, and $g_r = f_r$. The Laplace equation (5) decomposes forms in an appropriate manner. Now, the first component ϕ will be of interest. By use of (6), (9) and Stokes theorem, we obtain

$$\begin{aligned} -\delta_x(G_r f)(x) &= \int_\Omega d_y G_{r-1,r-1}(x,y) \wedge *_y f_r(y) = \\ &= \int_{\partial\Omega} G_{r-1,r-1}(x,y) \wedge *_y f_r(y) - \int_\Omega G_{r-1,r-1}(x,y) \wedge *_y \delta_y f_r(y) = -\phi(x)\,, \end{aligned} \tag{11}$$

where $\delta\phi = 0$. The equation for the second component θ will be proven in the same way. For this, we use (7), (10) and Stokes theorem to get

(12)
$$-d_x(G_r f)(x) = \int_{\Omega} \delta_y G_{r+1,r+1}(x,y) \wedge *_y f_r(y) =$$
$$= \int_{\partial\Omega} f_r(y) \wedge *_y G_{r+1,r+1}(x,y) - \int_{\Omega} d_y f_r(y) \wedge *_y G_{r+1,r+1}(x,y) = -\theta(x)\,,$$
where $d\theta = 0$.

□

As the only harmonic form on the simply connected manifold $\mathcal{M}$ is the null form, we state that $H\tilde{f} = 0$ for such manifolds.

Now, the local representation of our decomposition result and the announced decomposition result for Euclidean manifolds can be deduced.

REMARK 2: *Let $(.,.)_y$ be the pointwise Riemannian inner product for r-forms as given in Definition 1, and let $0 < r < n$. In order to derive local representations of the results (11) and (12), the following fomula*

(13)
$$(f, (\alpha, h)) = (\alpha \wedge f, h)\,,$$

for r-forms f, $(r+1)$-forms h and 1-forms α will be taken into account. By expressing the forms ϕ and θ locally, we obtain

$$\phi(x) = -\int_{\partial\Omega} (G_{r-1,r-1}(x,y), (\nu, f)(y))_y \, d\omega_y + \int_{\Omega} (G_{r-1,r-1}(x,y), \delta_y f_r(y))_y \, dy$$

and

$$\theta(x) = -\int_{\partial\Omega} (G_{r+1,r+1}(x,y), (\nu \wedge f)(y))_y \, d\omega_y + \int_{\Omega} (G_{r+1,r+1}(x,y), d_y f_r(y))_y \, dy\,,$$

where $dy := dV_y$.

Moreover, we obtain the convenient result

(14)
$$*(\alpha, f) = (-1)^{r-1} \alpha \wedge *f$$

from equation (13).

The fundamental theorem for $\mathbb{R}^n$-differential forms, where $n > 2$, is given in [9].

As stated in the fundamental theorem, the potential ϕ is coclosed. We will generalize ϕ to a coclosed form ξ. This will be useful to solve important boundary value problems.

LEMMA 2: *Let* $0 < r \leq n$, $g = g_{r-1} \in C^{1,\lambda}(\bar{\Omega})^{r-1}$, *where* $\delta g = 0$, *and* $f = f_r \in C^{1,\lambda}(\partial\Omega)^r$. *For the differential form* ξ, *defined by*

$$\xi(x) := -\int_{\partial\Omega} G_{r-1,r-1}(x,y) \wedge *_y f_r(y) + \int_{\Omega} G_{r-1,r-1}(x,y) \wedge *_y g_{r-1}(y)\,, \quad \textit{where } x \in \Omega, \tag{15}$$

the equation $\delta\xi = 0$ *is valid if and only if*

$$\delta(\nu, f) + (\nu, g) = 0 \quad \textit{on } \partial\Omega\,. \tag{16}$$

Proof: The property $\xi \in C^{2,\lambda}(\bar{\Omega})^{r-1}$ is shown by means of $\exp_x$ and regularity results for the Euclidean case. Furthermore, the coderived form $\delta\xi$ can be expressed as follows

$$\delta\xi(x) = -\int_{\partial\Omega} \delta_x G_{r-1,r-1}(x,y) \wedge *_y f_r(y) + \int_{\Omega} \delta_x G_{r-1,r-1}(x,y) \wedge *_y g_{r-1}(y)\,, \quad \text{where } x \in \Omega\,. \tag{17}$$

Since $\delta_x G_{r-1,r-1}(x,y) = 0$ for $r = 1$, we can now assume that $r > 1$. The boundary integral occuring in (17) will be transformed by means of (7), the property

$$d_y G_{r-2,r-2}(x,y) \wedge *_y f_r(y) = = d_y(G_{r-2,r-2}(x,y) \wedge *_y f_r(y)) \; + \; G_{r-2,r-2}(x,y) \wedge *_y \delta_y f_r(y)\,, \tag{18}$$

and

$$\delta(\nu, f) = -(\nu, \delta f)\,. \tag{19}$$

Equation (19) can be inferred by using admissable boundary coordinate systems as defined in [10, p. 300]. We obtain

$$\int_{\partial\Omega} \delta_x G_{r-1,r-1}(x,y) \wedge *_y f_r(y) = -\int_{\partial\Omega} G_{r-2,r-2}(x,y) \wedge *_y \delta_y f_r(y) = = \int_{\partial\Omega} (G_{r-2,r-2}(x,y), \delta_y(\nu, f_r)(y))_y\,. \tag{20}$$

As δg vanishes, equation (7) delivers

$$\delta_x G_{r-1,r-1}(x,y) \wedge *_y g_{r-1}(y) = -d_y(G_{r-2,r-2}(x,y) \wedge *_y g_{r-1}(y))\,. \tag{21}$$

Therefore we conclude

$$\delta\xi(x) = -\int_{\partial\Omega} (G_{r-2,r-2}(x,y), \delta_y(\nu, f_r)(y) + (\nu, g_{r-1})(y))_y \, d\omega_y \,. \tag{22}$$

Referring to the well-known jump properties for derivatives of single layer potentials, we state that $\delta(\nu, f)(y) + (\nu, g)(y) = 0$ if $\delta\xi = 0$.

□

Differential forms like the presented form ξ are useful to find solutions for some boundary value problems (cf. [8, 9, 12]). Equation

$$\delta(\nu, f)(y) + (\nu, g)(y) = 0$$

is one of the integrability conditions.

4 Harmonic Fields on Compact Riemannian Manifolds

This section shall investigate Dirichlet and Neumann fields. In order to achieve this, we generalize the integral equations given in [9].

DEFINITION 3: *Let $0 < r \leq n$. The integral operator K_{r-1} is defined by*

$$K_{r-1} = K_{k,\lambda,r-1} : C_\nu^{k,\lambda}(\partial\Omega)^{r-1} \longrightarrow \mathcal{R}(K_{k,\lambda,r-1}),$$

$$(K_{r-1}\epsilon)(x) := -2 \int_{\partial\Omega} (\nu(x), (d_x G_{r-1,r-1}(x,y), \epsilon(y))_y) \, d\omega_y \,.$$

We are able to show a number of important properties of K_{r-1} using local charts.

REMARK 3: *For the Euclidean manifold $\mathbf{R}^n$, the operator K_{r-1} can be represented by*

$$(K_{r-1}\epsilon)(x) := -2 \int_{\partial\Omega} (\nu(x), (d_x G_0(x,y) \wedge \epsilon(y))) \, d\omega_y \,.$$

For $n > 2$, the operator K_{r-1} corresponds with the one of [4, 5, 9].

Now, we are interested in the regularity of $K_{r-1}\epsilon$ and in properties of the adjoint as well as the Riesz number of K_{r-1}.

LEMMA 3: *The operator $K_{k,\lambda,r-1}$, where $0 < r \leq n$, belongs to*

$$\mathcal{L}(C_\nu^{k,\lambda}(\partial\Omega)^{r-1}, C_\nu^{k+1,\lambda}(\partial\Omega)^{r-1}).$$

Proof: After transformation by means of $\exp_x$, the properties of the relevant kernels in $\mathbb{R}^n$, endowed with Euclidean metric, have to be considered. Then we proceed as in the proof of *Lemma 1*. Furthermore, we refer to results of [3, 4, 5]. Although the case $n = 2$ is not explicitly included there, it is not difficult to adapt the proofs appropriately.

□

The exterior derivative of the single layer potential is discontinuous in the normal vector's direction. This will be shown in the following Lemma.

LEMMA 4: *Let $0 < r \leq n$, $f \in C^{0,\lambda}(\bar{\Omega})^r$ or $f \in C^{0,\lambda}(\bar{\tilde{\Omega}})^r$, respectively, and $\epsilon = -(\nu, f)$. For the differential form u, defined by*

$$u(x) = u_{r-1}(x) := \int_{\partial\Omega} G_{r-1,r-1}(x,y) \wedge *_y f(y) = -\int_{\partial\Omega} (G_{r-1,r-1}(x,y), \epsilon(y))\, d\omega_y\,,$$

the relation

$$du_{\mp}(x) = \int_{\partial\Omega} d_x G_{r-1,r-1}(x,y) \wedge *_y f(y) \mp \frac{1}{2}(\nu \wedge (\nu, f))(x)\,, \text{ where } x \in \partial\Omega\,,$$

is valid. The upper index "−" relates to interior approximations and the lower one "+" to exterior approximations.

Proof: We set $p_l := \exp_x^{-1} x_l$ and $q := \exp_x^{-1} y$, as we may confine ourselves to points in the neighbourhood of x. In the proof of *Lemma 1*, the decomposition

$$G(x,y) = \omega(x,y) + W(x,y) \tag{23}$$

was used, where ω shows "Newton-type" singularities, and W possesses weaker ones.

After transformation by means of exp, $W(x,y)$ is mapped to a double form which components are $\pm\tilde{W}_0(p,q)$. Since $\tilde{W}_0(p,q)$ will not affect the regarded properties, we will proceed with calculations for $\omega(x,y)$.

Then the well-known equation

$$\partial_{p^i} \lim_{p_l \to p} \int_{\partial\tilde{\Omega}} \tilde{\omega}_0(p_l, q)\tilde{\epsilon}(q)\, d\omega_q = \tag{24}$$

$$= \lim_{p_l \to p} \partial_{p^i} \int_{\partial\tilde{\Omega}} \tilde{\omega}_0(p_l, q)\tilde{\epsilon}(q)\, d\omega_q \pm \frac{1}{2}\tilde{\nu}^i(p)\tilde{\epsilon}(p)$$

for the differential form $\tilde{\epsilon}$ is taken into account. Here p^i and ν^i denote Euclidean components of p or ν, respectively. The upper sign "+" is used for an interior approximation and the lower one "−" for an exterior approximation.

Finally the equations (2) and (24) yield

$$d \lim_{p_l \to p} \int_{\partial\tilde{\Omega}} (\tilde{G}_{r-1,r-1}(p_l, q), \tilde{\epsilon}(q))_q \, d\omega_q = \tag{25}$$

$$= \lim_{p_l \to p} d \int_{\partial\tilde{\Omega}} (\tilde{G}_{r-1,r-1}(p_l, q), \tilde{\epsilon}(q))_q \, d\omega_q \pm \frac{1}{2}(\tilde{\nu} \wedge \tilde{\epsilon})(p) .$$

By means of this equation, the assertion is confirmed.

□

For boundary value problems of this and the next chapter the following constraint will be important:

$$\langle \epsilon, \gamma_i \rangle = Y_i \quad \text{for all} \quad \gamma_i \in \mathcal{N}(I - K^+_{r-1}) ,$$

where K^+_{r-1} denotes the adjoint of K_{r-1} with regard to the bilinear form $\langle .,. \rangle$ given in *Definition 1.* Now, we will show that the index

$$n_1 := \max\{n \in \mathbb{N} \,|\, \mathcal{N}(I - K_{r-1})^{n-1} \neq \mathcal{N}(I - K_{r-1})^{n-1}\}$$

is equal to 1. Thus the mentioned constraint can be fulfilled.

<u>LEMMA 5:</u> *The Riesz number of the operator K_{r-1}, $0 < r \leq n$, equals 1.*

Proof. We will generalize the approach of [9]. Let $\epsilon^{(i)} \in C^{0,\lambda}_{\nu}(\partial\Omega)^{r-1}$, where $i = 0, 1$, be solutions of

$$(I - K_{r-1})\epsilon^{(0)} = 0 , \quad (I - K_{r-1})\epsilon^{(1)} = \epsilon^{(0)} . \tag{26}$$

We set $\epsilon^{(i)} =: -(\nu, f^{(i)})$, where $f^{(i)} \in C^{0,\lambda}(\partial\Omega)^r$.

Let furthermore

$$\xi^{(i)}(x) := \int_{\partial\Omega} (G_{r-1,r-1}(x, y), \epsilon^{(i)}(y)) \, d\omega_y = \tag{27}$$

$$= -\int_{\partial\Omega} G_{r-1,r-1}(x, y) \wedge *f^{(i)}_r(y) , \quad x \in \mathcal{M} \backslash \partial\Omega .$$

According to equations (26), the forms $\epsilon^{(i)}$ are elements of $C^{k,\lambda}(\partial\Omega)^{r-1}$ for arbitrary $k \in \mathbb{N}_0$. Therefore the forms $\xi^{(i)}$ belong to $C^{k+1,\lambda}(\bar{\Omega})^{r-1}$.

The case $r = 1$ provides 0-forms $\epsilon^{(1)}$ and 0-forms $\xi^{(1)}$. This implies that $\delta\xi^{(1)}$ vanishes.

For the non-trivial cases $r > 1$, we use property (6) to show

$$\delta\xi^{(1)}(x) = -\int_{\partial\Omega} \delta_x G_{r-1,r-1}(x,y) \wedge *f_r^{(1)}(y) = \tag{28}$$

$$= \int_{\partial\Omega} d_y G_{r-2,r-2}(x,y) \wedge *f_r^{(1)}(y) = \int_{\partial\Omega} G_{r-2,r-2}(x,y) \wedge *\delta f_r^{(1)}(y)\,.$$

The differential forms $\xi^{(i)}$ are harmonic forms in $\mathcal{M}\backslash\partial\Omega$, as can be proven by elementary methods.

We obtain from *Lemma 4* and (28)

$$\nu \wedge \delta d\xi_+^{(1)} - \nu \wedge \delta d\xi_-^{(1)} = 0\,. \tag{29}$$

By this Lemma, a further jump property is given:

$$d\xi_+^{(0)} - d\xi_-^{(0)} = \nu \wedge \epsilon^{(0)}\,. \tag{30}$$

Equation (26) delivers

$$2(\nu, d\xi_+^{(0)}) = -K_{r-1}\epsilon^{(0)} + (\nu, \nu \wedge \epsilon^{(0)}) = (I - K_{r-1})\epsilon^{(0)} = 0\,. \tag{31}$$

We conclude from property $\Delta\xi^{(0)} = 0$ that $d\delta d\xi^{(0)}$ vanishes in $\mathcal{M}\backslash\partial\Omega$. Equation (31) then delivers $\delta d\xi^{(0)}|_{\hat{\Omega}} = 0$ and $d\xi^{(0)}|_{\hat{\Omega}} = 0$. Therefore (30) implies

$$-d\xi_-^{(0)} = d\xi_+^{(0)} - d\xi_-^{(0)} = \nu \wedge \epsilon^{(0)}\,. \tag{32}$$

By means of the property

$$2(\nu, d\xi_+^{(1)}) = (I - K_{r-1})\epsilon^{(1)} = \epsilon^{(0)} \tag{33}$$

and (32), we infer that

$$2(\nu, d\xi_+^{(1)}) + (\nu, d\xi_-^{(0)}) = 0\,. \tag{34}$$

The results $d\delta d\xi^{(1)} = 0$, (34) and (29) are taken into account in order to state

$$\int_{\hat{\Omega}} (\delta d\xi^{(1)}, \delta d\xi^{(1)})\, dx = -\frac{1}{2}\int_{\Omega} (\delta d\xi^{(1)}, \delta d\xi^{(0)})\, dx\,. \tag{35}$$

Referring to the analogous equation to (29), formulated for $\xi^{(0)}$, (32) and $d\xi^{(0)}|_{\hat{\Omega}} = 0$, we summarize that

$$\nu \wedge d\xi_-^{(0)} = 0\,,\quad -(\nu, d\xi_-^{(0)}) = \epsilon^{(0)} \text{ and } \nu \wedge \delta d\xi_-^{(0)} = 0\,. \tag{36}$$

The form $\delta d\xi^{(0)}|_{\Omega}$ vanishes, as the boundary value $\nu \wedge \delta d\xi_-^{(0)}$ vanishes. By means of (35), we thus obtain $\delta d\xi^{(1)}|_{\hat{\Omega}} = 0$. Moreover, the first equation of (36) entails that the form $d\xi^{(0)}|_{\Omega}$ is a Dirichlet field.

As $u_-^{(0)}$ equals $u_+^{(0)}$, we obtain from (34):

$$2((\nu, d\xi_+^{(1)}), \xi_+^{(0)}) + ((\nu, d\xi_-^{(0)}), \xi_-^{(0)}) = 0\,. \tag{37}$$

This equation, $\delta d\xi^{(0)}|_\Omega = 0$ and $d\xi^{(0)}|_{\hat{\Omega}} = 0$ together imply that $d\xi^{(0)}|_\Omega = 0$. Finally the second equation of (36) shows that $\epsilon^{(0)}$ vanishes.

□

In order to use results from Fredholm theory, we are interested in properties of the adjoint of the operator K_{r-1}.

<u>DEFINITION 4:</u> *Let $0 \leq r < n$. The integral operator L_{r+1} is defined by*

$$L_{r+1} = L_{k,\lambda,r+1} : C_\tau^{k,\lambda}(\partial\Omega)^{r+1} \longrightarrow \mathcal{R}(L_{k,\lambda,r+1}),$$

$$(L_{r+1}\gamma)(x) := -2 \int\limits_{\partial\Omega} (\nu(x) \wedge (\delta_x G_{r+1,r+1}(x,y), \gamma(y))_y)\, d\omega_y\,.$$

The operator L_{r+1} for the Euclidean manifold $\mathbf{R}^n$, $n > 2$, [4, 5, 9]. Now, we will show that this particular representation can be derived from our general definition.

<u>REMARK 4:</u> *For the Euclidean manifold $\mathbf{R}^n$, the operator L_{r+1} may be represented by*

$$(L_{r+1}\gamma)(x) := -2 \int\limits_{\partial\Omega} (\nu(x) \wedge (dG_0(x,y), \gamma(y)))\, d\omega_y\,.$$

We will notice that application of L improves regularity, in accordance to the regularity results for the operator K. This can be proven in the same manner as above.

<u>LEMMA 6:</u> *The operator $L_{k,\lambda,r+1}$, where $0 \leq r < n$, belongs to*

$$\mathcal{L}(C_\tau^{k,\lambda}(\partial\Omega)^{r+1}, C_\tau^{k+1,\lambda}(\partial\Omega)^{r+1}).$$

With regard to the bilinear form of *Definition 1 c)*, the operator $-L_{r+1}$ is the adjoint of our previously defined operator K_r, i.e. $K_r^+ = -L_{r+1}$. Each operator L equals a composition of a operator K and Hodge operators. This offers the possibility to express Neumann problems for r-forms as Dirichlet problems for $(n-r)$-forms, where $0 < r < n$.

<u>LEMMA 7:</u> *Let $0 \leq r < n$. The operator $-L_{r+1} \in \mathcal{L}(L_\tau^2(\partial\Omega)^{r+1}, L_\tau^2(\partial\Omega)^{r+1})$ is the adjoint of $K_r \in \mathcal{L}(L_\nu^2(\partial\Omega)^r, L_\nu^2(\partial\Omega)^r)$ with regard to the bilinear form $\langle .,. \rangle$ given in Definition 1 c).*

Proof: Let u, v be 1-forms and f be an l-form, where $0 \leq l \leq n$. Then

$$f(u,v) = u \wedge (v, f) + (v, u \wedge f)\,. \tag{38}$$

The differential forms ϵ and γ are chosen according to $\epsilon \in L^2_\nu(\partial\Omega)^r$ and $\gamma \in L^2_\tau(\partial\Omega)^{r+1}$. For the relevant term $\langle K_r\epsilon_r, \gamma_{r+1}\rangle$, we may write

$$\frac{1}{2}\langle K_r\epsilon_r, \gamma_{r+1}\rangle = \tag{39}$$

$$= -\int_{\partial\Omega}\int_{\partial\Omega}(\nu(x) \wedge (\nu(x), (d_xG_{r,r}(x,y), \epsilon_r(y)), \gamma_{r+1}(x))\, d\omega_y d\omega_x\,.$$

Decomposition (38) delivers the representation

$$\nu(x) \wedge (\nu(x), (d_xG_{r,r}(x,y), \epsilon_r(y)) = \tag{40}$$

$$= (d_xG_{r,r}(x,y), \epsilon_r(y)) - (\nu(x), \nu(x) \wedge (d_xG_{r,r}(x,y), \epsilon_r(y))\,.$$

It is required that $\nu \wedge \gamma = 0$. Hence

$$\frac{1}{2}\langle K_r\epsilon_r, \gamma_{r+1}\rangle = -\int_{\partial\Omega}\int_{\partial\Omega}((d_xG_{r,r}(x,y), \epsilon_r(y)), \gamma_{r+1}(x))\, d\omega_y d\omega_x\,. \tag{41}$$

We derive from equation (7) and (38)

$$\frac{1}{2}\langle K_r\epsilon_r, \gamma_{r+1}\rangle = \tag{42}$$

$$= \int_{\partial\Omega}\int_{\partial\Omega}(\nu(y) \wedge \epsilon_r(y), \nu(y) \wedge (\delta_yG_{r-1,r+1}(x,y), \gamma_{r-1}(x))\, d\omega_y d\omega_x\,.$$

As stated in [11, p. 133], the kernel $G_{r+1,r+1}(x,y)$ is symmetric. Therefore $\frac{1}{2}\langle K_r\epsilon_r, \gamma_{r+1}\rangle$ equals

$$\int_{\partial\Omega}(\nu(x) \wedge \epsilon_r(x), (\int_{\partial\Omega} \nu(x) \wedge (\delta_xG_{r+1,r+1}(x,y), \gamma_{r+1}(y))\, d\omega_y)d\omega_x = \tag{43}$$

$$= -\frac{1}{2}\langle \epsilon_r, L_{r+1}\gamma_{r+1}\rangle\,.$$

□

One of the constraints of the mentioned problem might be characterized as a topological one, since it is directly related to homological properties. This constraint $Y_i = \langle \epsilon, \gamma_i\rangle$ can be formulated as follows:

We will find Neumann fields $\hat{z}_i \in \mathcal{Z}(\hat{\Omega})^{r-1}$ fulfilling

$$\gamma_i = \nu \wedge \hat{z}_i \ \text{ for each } \gamma_i \in \mathcal{N}(I - K^{\top}_{r-1})\,.$$

On the other hand, ϵ is given as $\epsilon = -(\nu, y)$, where $y \in \mathcal{Y}(\Omega)^r$. Referring to (13) and the decomposition (38), we can write

$$Y_i = \langle \epsilon, \gamma_i \rangle =$$

$$= \int_{\partial\Omega} (\epsilon, (\nu, \gamma_i))\, d\omega = - \int_{\partial\Omega} (y, \nu \wedge \hat{z}^i)\, d\omega\,.$$

The following result can be derived by means of the commutation rule $*G = G*$ given in [11, p. 134].

<u>LEMMA 8:</u> *Let* $0 < r \leq n$. *The operator* $*L_{n-(r-1)}$ *equals* $K_{r-1}*$.

Now, we deal with the announced harmonic fields. Since each $(n-r)$-Neumann field can be mapped to an r-Dirichlet field, we may confine ourselves to Dirichlet fields.

<u>THEOREM 2:</u> *We presuppose that* $0 < r < n$. *The form* $f \in C^{1,\lambda}(\bar{\Omega})^r$ *is extended by zero to the form* $\tilde{f} \in L^2(\mathcal{M})^r$.

a) Let the differential form $f \in C^{1,\lambda}(\bar{\Omega})^r$ *be a Dirichlet field, where* $H\tilde{f} = 0$. *Then*

$$\epsilon := -(\nu, f)$$

is a solution of the homogeneous equation

$$(I - K_{r-1})\epsilon = 0\,. \tag{44}$$

b) If ϵ *is a solution of this equation, then the r-form*

$$f(x) = f_r(x) := d \int_{\partial\Omega} (G_{r-1,r-1}(x,y), \epsilon_{r-1}(y))_y\, d\omega_y \tag{45}$$

is a $C^{k,\lambda}(\bar{\Omega})^r$*-form on* Ω, *and a Dirichlet field on* Ω, *where the values* $k \in \mathbb{N}$ *and* $0 < \lambda < 1$ *are arbitrary. Moreover,* f *has the properties*

$$-(\nu, f) = \epsilon \tag{46}$$

and

$$Hf = 0 \ \ on\ \mathcal{M}\,. \tag{47}$$

Proof: Kress's arguments presented in [9] and *Theorem 1* will be used.

a) For the presupposed Dirichlet field f we thus conclude that

$$f(x) = f_r(x) = d\xi_{r-1}(x)\,, \quad x \in \Omega\,, \tag{48}$$

where the potential ξ_{r-1} is given by

(49)
$$\xi_{r-1}(x) = -\int_{\partial\Omega} G_{r-1,r-1}(x,y) \wedge *_y f_r(y) =$$
$$= \int_{\partial\Omega} (G_{r-1,r-1}(x,y), \epsilon(y))_y \, d\omega_y .$$

Therefore the assertion (44) is proven by means of *Lemma 4*.

b) The regularity result $\epsilon \in C^{k,\lambda}(\partial\Omega)^{r-1}$ is inferred from the homogeneous integral equation and the regularity for the operator K_{r-1}, cf. *Lemma 3*. According to the assumption (45) we can write

(50)
$$f(x) = f_r(x) := d\xi_{r-1}(x) ,$$

where

$$\xi_{r-1}(x) := \int_{\partial\Omega} (G_{r-1,r-1}(x,y), \epsilon(y))_y \, d\omega_y .$$

As ϵ belongs to $C^{k,\lambda}(\partial\Omega)^{r-1}$, the differential form ξ is in $C^{k+1,\lambda}(\partial\Omega)^{r-1}$. Furthermore, an r-form $f' \in C^{k,\lambda}(\partial\Omega)^r$ exists which fulfils $-(\nu, f') = \epsilon$.

Since ξ_{r-1} is a harmonic form in $\mathcal{M}\backslash\partial\Omega$, the equation

(51)
$$\delta f = \delta d\xi = -d\delta\xi$$

is valid in $\mathcal{M}\backslash\partial\Omega$. As already used above, we write the index "$-$" for the interior approximations and the index "$+$" for the exterior ones. Conclusions from *Lemma 4* and the presumed integral equation are

(52)
$$(\nu, f_+) = 0$$

and

(53)
$$f_+ - f_- = \nu \wedge \epsilon .$$

For the coderived form $\delta\xi$, the equation

(54)
$$\delta\xi_{r-1}(x) = -\int_{\partial\Omega} \delta_x G_{r-1,r-1}(x,y) \wedge *_y f_r'(y)$$

is given. Since this form vanishes for $r = 1$, we may now confine ourselves to $2 \leq r < n$. Analogously to (28) the form $\delta\xi_{r-1}$ can be suitably written such that

(55)
$$\delta\xi_{r-1}(x) = \int_{\partial\Omega} G_{r-2,r-2}(x,y) \wedge *_y \delta_y f_r'(y) .$$

Then the representation of $d\delta\xi$, *Lemma 4* and equation (51) yield

$$\nu \wedge \delta f_+ - \nu \wedge \delta f_- = 0 . \tag{56}$$

Finally the assertion is proven using the well-known arguments of [9].

□

REMARK 5: *Let $0 < r < n$ and let $B_r = B_r(\Omega)$ be the Betti number of order r with regard to the set Ω.*

a) *The preceding Theorem 2 can be formulated for Neumann fields as well. This can be shown by means of the Hodge mapping $*$.*

b) *There exist a unique Dirichlet field on Ω which satisfies*

$$-\int_{\partial\Omega} (y, \nu \wedge \hat{z}^i)\, d\omega = Y_i , \quad \textit{for all } \hat{z}_i \in \mathcal{Z}(\hat{\Omega})^{r-1}, \qquad i = 1, ..., B_{n-r} . \tag{57}$$

This is a consequence of Theorem 2 and the issue to follow.

c) *Lemma 5 shows that there is a uniquely defined solution ϵ of $\mathcal{N}(I - K_{r-1})$ which fulfils*

$$\langle \epsilon, \gamma_i \rangle = Y_i , \quad \textit{for all } \gamma_i \in \mathcal{N}(I - K^+_{r-1}) , i = 1, ..., B_{n-r} . \tag{58}$$

As stated in [7], the number of linear independent Dirichlet (Neumann) r-fields on Ω, where $0 < r < n$, is equal to the Betti number $B_{n-r} = B_{n-r}(\Omega)$ ($B_r = B_r(\Omega)$). Generalizing the results of [9], we have shown that

$$dim\, \mathcal{N}(I - K_{r-1}) = B_{n-r}(\Omega) \tag{59}$$

and

$$dim\, \mathcal{N}(I - L_{r+1}) = B_r(\Omega). \tag{60}$$

Furthermore, Alexander's duality theorem points out the relationship of Betti numbers for Ω and $\hat{\Omega}$.

5 Dirichlet and Neumann Boundary Problems

Our main result of this chapter is stated in *Theorem 3*. There the boundary value problem is connected with a Fredholm problem for differential forms defined on the boundary $\partial\Omega$. The concerning boundary problem, called Dirichlet Problem, will be presented in the subsequent section.

DEFINITION 5: Dirichlet Problem

Let $0 < r < n$, $h \in C^{1,\lambda}(\bar{\Omega})^{r+1}$, $g \in C^{1,\lambda}(\bar{\Omega})^{r-1}$ *and* $\gamma \in C_\tau^{1,\lambda}(\partial\Omega)^{r+1}$. *Moreover,* $\{\hat{z}_i\}_{i=1,\dots,B_{n-r}}$ *is a basis of* $\mathcal{Z}(\hat{\Omega})^{r-1}$ *and* Y_i *are real numbers. Then a solution* $f \in C^{1,\lambda}(\bar{\Omega})^r$ *of the Dirichlet problem has to fulfil the local equations*

$$df = h \ \ and \ \ \delta f = g \ \ in\,\Omega \qquad (61)$$
$$-\nu \wedge f = \gamma \ \ on\,\partial\Omega$$

and the topological constraint

$$-\int_{\partial\Omega} (f, \nu \wedge \hat{z}_i)\, d\omega = Y_i, \ \ i = 1, \dots, B_{n-r}\,. \qquad (62)$$

Evidently there exist integrability conditions of this boundary value problem. Later we will realize that the below listed necessary conditions are sufficient ones.

LEMMA 9: Integrability Conditions for the Dirichlet Problem

Let $0 < r < n$ *and let the assumptions of Definition 5 be fulfilled. Then the necessary conditions (63)-(67) are given as follows: We presuppose that*

$$dh = 0 \ \ in\,\Omega \qquad (63)$$

and

$$\delta g = 0 \ \ in\,\Omega\,. \qquad (64)$$

Both forms h and g, extended by zero to $L^2(\mathcal{M})$*-forms, are required to be elements of* $H^\perp L^2(\mathcal{M})^s$, *where s equals* $r+1$ *or* $r-1$, *respectively. These extended forms are called h and g again.*

Moreover, we presuppose that

$$d\gamma - \nu \wedge h = 0 \ \ on\,\partial\Omega\,, \qquad (65)$$

$$\int_{\Omega} (h, y)\, dx + \int_{\partial\Omega} (\gamma, y)\, d\omega = 0 \ \ for\ all\, y \in \mathcal{Y}(\Omega)^{r+1}\,, \qquad (66)$$

and

(67)
$$\int_\Omega (g, y)\, dx = 0 \quad \text{for all } y \in \mathcal{Y}(\Omega)^{r-1}\,.$$

For $r = 1$, the conditions (64) and (67) are redundant, for $r = n-1$ the conditions (63) and (65) can be omitted.

<u>THEOREM 3:</u> *We assume that $0 < r < n$, $h \in C^{1,\lambda}(\bar\Omega)^{r+1}$, $g \in C^{1,\lambda}(\bar\Omega)^{r-1}$ and $\gamma \in C_\tau^{1,\lambda}(\partial\Omega)^{r+1}$. The form $f \in C^{1,\lambda}(\bar\Omega)^r$ is extended by zero to the form $\tilde f \in L^2(\mathcal{M})^r$.*

a) *Let $f \in C^{1,\lambda}(\bar\Omega)^r$ be a solution of the Dirichlet problem which satisfies $H\tilde f = 0$. Then $\epsilon(x) := -(\nu, f)(x)$ is a solution of the inhomogeneous integral equation*

(68)
$$(I - K_{r-1})\epsilon = \mu(h, g, \gamma)\,, \quad \text{where}$$

$$\mu(h,g,\gamma)(x) := -2\,(\nu(x), d\int_\Omega G_{r-1,r-1}(x,y) \wedge *_y g_{r-1}(y) +$$
$$\delta[\int_\Omega h_{r+1}(y) \wedge *_y G_{r+1,r+1}(x,y) \; + \int_{\partial\Omega} (G_{r+1,r+1}(x,y), \gamma_{r+1}(y))_y\, d\omega_y])\,,$$
$$\text{for } x \in \partial\Omega\,,$$

and the topological constraint

(69)
$$\langle \epsilon, \nu \wedge \hat z_i \rangle = Y_i, \quad i = 1, \ldots, B_{n-r},$$

is fulfilled.

b) *Vice versa, if ϵ is a solution of the integral problem (68) and (69), and the integrability conditions of Lemma 9 are satisfied, then*

(70)
$$f := d\xi + \delta\zeta\,,$$

where

$$\xi(x) := \int_{\partial\Omega} (G_{r-1,r-1}(x,y), \epsilon_{r-1}(y))_y\, d\omega_y + \int_\Omega G_{r-1,r-1}(x,y) \wedge *_y g_{r-1}(y)$$

and

$$\zeta(x) := \int_{\partial\Omega} (G_{r+1,r+1}(x,y), \gamma_{r+1}(y))_y\, d\omega_y + \int_\Omega h_{r+1}(y) \wedge *_y G_{r+1,r+1}(x,y)\,,$$

is the uniquely determined $C^{1,\lambda}(\bar\Omega)^r$-solution of the Dirichlet problem on Ω. Additionally, the solution f shows the property $Hf = 0$ on $\mathcal{M}$.

Proof: Essential parts of the proof are similar to the corresponding ones in [9] and [4]. On the other hand, some arguments differ. Therefore we will also repeat some known context. Since part a) of the assertion is a consequence of *Theorem 1* and *Lemma 4*, it merely remains to prove part b) of the equivalence.

The components of the differential forms ξ and ζ belong to $C^{2,\lambda}(\bar{\Omega})$ or $C^{2,\lambda}(\bar{\hat{\Omega}})$, respectively. As the extended forms h and g are elements of $H^{\perp}L^2(\mathcal{M})^s$, where s equals $r+1$ or $r-1$, this implies

(71) $$\Delta\xi = 0 \text{ and } \Delta\zeta = 0 \text{ in } \hat{\Omega}$$

(72) $$\Delta\xi = g \text{ and } \Delta\zeta = h \text{ in } \Omega .$$

By means of equations (71) and (72) we derive convenient expressions:

(73) $$\delta f = -d\delta\xi \text{ in } \hat{\Omega}$$

(74) $$\delta f = g - d\delta\xi \text{ in } \Omega .$$

Lemma 2 and integrability condition (65) entail that $d\zeta = 0$ in Ω as well as in $\hat{\Omega}$. The differential form df thus equals $\Delta\zeta$. Then equations (71) and (72) show that

(75) $$df = 0 \quad \text{in } \hat{\Omega} \quad \text{and} \quad df = h \quad \text{in } \Omega .$$

Lemma 4 delivers

(76) $$d\xi_+ - d\xi_- = \nu \wedge \epsilon \quad \text{and} \quad \delta\theta_+ - \delta\theta_- = (\nu, \gamma) \quad \text{on } \partial\Omega .$$

This can be summarized as jump property for f:

(77) $$f_+ - f_- = \nu \wedge \epsilon + (\nu, \gamma) \quad \text{on } \partial\Omega .$$

Thus integral equation (68) implies that (ν, f_+) vanishes.

Arguing as for the proof of *Lemma 2*, we obtain

(78) $$\delta\xi(x) = \int_{\partial\Omega} (G_{r-2,r-2}(x,y), \delta_y \epsilon_{r-1}(y) - (\nu, g_{r-1})(y))_y \, d\omega_y .$$

Therefore the form $\nu \wedge d\delta\xi$ is continuous at $\partial\Omega$.

The form $f_{|\hat{\Omega}}$ is an element of $\mathcal{Z}(\hat{\Omega})^r$, which can be derived from $d\delta f_{|\hat{\Omega}} = 0$, $(\nu, f_+) = 0$ and the first equation of (75). Moreover, we obtain that

(79) $$\int_{\partial\Omega} (f_+, (\nu, y)) \, d\omega = 0 \quad \text{for all } y \in \mathcal{Y}(\Omega)^{r+1} ,$$

by virtue of (65) and (77). The form f thus vanishes in $\hat{\Omega}$.

According to (64) and (74) we infer that $\delta d\delta\xi_{|\Omega} = 0$. As $f_{|\bar{\hat{\Omega}}}$ vanishes, the form $d\delta\xi_+$ vanishes too. We have stated in (29) that $\nu \wedge d\delta\xi$ is continuous at $\partial\Omega$. This implies that $\nu \wedge d\delta\xi_- = 0$.

Therefore $d\delta\xi|_{\Omega}$ is an element of $\mathcal{Y}(\Omega)^{r-1}$, and we conclude, by means of the integrability condition (67):

$$\int_{\Omega} (d\delta\xi, d\delta\xi)\, dx = -\int_{\Omega} (d\delta\xi, \delta f)\, dx = -\int_{\partial\Omega} (\nu \wedge d\delta\xi_{-}, f)\, d\omega = 0\,. \tag{80}$$

Hence the form $\delta\xi$ vanishes on Ω.

The properties $f|_{\tilde{\Omega}} = 0$ and (77) entail that

$$-\nu \wedge f_{-} = \gamma \quad \text{and} \quad -(\nu, f_{-}) = \epsilon\,. \tag{81}$$

Finally we have to show that the inhomogeneous integral equation (68) is solvable. This is the case if

$$\langle \mu, \gamma_i \rangle = 0 \quad \text{for all}\ \gamma_i \in \mathcal{N}(I - K^{+}_{r-1}) \tag{82}$$

is valid for $\mu = \mu(h, g, \gamma)$. The elements of $\mathcal{N}(I - K^{+}_{r-1})$ can be expressed as differential forms $\nu \wedge \hat{z}_i$, where $(\hat{z}_i)$ is the basis of Neumann (r-1)-fields on $\hat{\Omega}$. Then some calculation delivers

$$\langle \mu, \gamma_i \rangle = \int_{\partial\Omega} (\mu, \hat{z}_i)\, d\omega = \tag{83}$$

$$= 2 \int_{\hat{\Omega}} (\delta d \int_{\Omega} G_{r-1,r-1}(x,y) \wedge *_y g_{r-1}(y), \hat{z}_i(x))\, dx =$$

$$= -2 \int_{\hat{\Omega}} (d\delta \int_{\Omega} G_{r-1,r-1}(x,y) \wedge *_y g_{r-1}(y), \hat{z}_i(x))\, dx =$$

$$= 2 \int_{\partial\Omega} (\delta \int_{\Omega} G_{r-1,r-1}(x,y) \wedge *_y g_{r-1}(y), (\nu, \hat{z}_i)(x))\, d\omega_x = 0\,,$$

where we have used that $Hg|_{\hat{\Omega}}$ vanishes.

Therefore the according Fredholm equation possesses a solution, and the solution is unique according to *Remark 5.*

□

The dual boundary value problem, denoted as Neumann Problem, can be derived from the Dirichlet Problem by using the Hodge mapping.

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