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A PRIORI ESTIMATES FOR DIFFERENTIAL FORMS WITH COMPONENTS IN $C^{1,\lambda}$

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Abstract: In this paper we prove optimal a priori estimates and regularity results for boundary value problems regarding differential forms. This concerns inequalities with respect to the $C^{1,\lambda}$- and $W^{1,q}$-norms. Particularly, we treat estimates of the gradient of differential forms with euclidian components in $C^{1,\lambda}$ by their exterior derivative, coderivative, suitable boundary values and values depending on the Betti numbers of the given area.

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1 Introduction

Our main objective is to give a priori ineqalities and regularity results for differential forms of order r with euclidian components in $C^{1,\lambda}$. In detail, we present appropriate estimates to boundary value problems for differential forms with respect to bounded sets $G \subset \mathbb{R}^n$, $n > 2$. These boundary value problems determine the exterior derivative, the codifferential, either the tangential or normal component of our differential forms and some quantities due to the topology of G. Special cases of such a priori inequalities play an essential role in connection with decomposition results (cf. [10], (Theorem 7.7.2)). Here we work on problems like those in [9], which in essence have been discussed in [4]. For vector fields, we take numerous results from [8] and [14]. Estimates for various spaces of vector fields in $\mathbb{R}^3$ are given e.g. in [1], [2] and, assuming special topological and boundary conditions, in [16].

The results of [16] and [2] will now be extended in relation to the degree of the discussed tensors. In addition, we slightly generalize the former inequalities by presenting estimates for derivatives of forms and for the forms themselves. Assuming f is a differential form of order r, $r \in \mathbb{N} : r < n$, comprising euclidian components in $C^{1,\lambda}$, we infer the estimates

$$\|f\|_{C^{1,\lambda}(\bar{G})} \leq c_1 \left(\|df\|_{C^{0,\lambda}(\bar{G})} + \|\delta f\|_{C^{0,\lambda}(\bar{G})} + \|\nu \wedge f\|_{C^{1,\lambda}(\partial G)} + \sum_{i=1}^{B_{n-r}} |Y_i|\right)$$

and

$$\|f\|_{C^{1,\lambda}(\bar{G})} \leq c_2 \left(\|df\|_{C^{0,\lambda}(\bar{G})} + \|\delta f\|_{C^{0,\lambda}(\bar{G})} + \|(\nu, f)\|_{C^{1,\lambda}(\partial G)} + \sum_{i=1}^{B_r} |Z_i|\right),$$

where $c_i = c_i(n,r,\lambda,G)$, $i = 1,2$, are positive constants. Moreover, the analogous L^q-estimates are valid for f, i. e.

$$\|f\|_{W^{1,q}(G)} \leq c_3 \left(\|df\|_{L^q(G)} + \|\delta f\|_{L^q(G)} + \|\nu \wedge f\|_{W^{1-\frac{1}{q},q}(\partial G)} + \sum_{i=1}^{B_{n-r}} |Y_i|\right)$$

and

$$\|f\|_{W^{1,q}(G)} \leq c_4 \left(\|df\|_{L^q(G)} + \|\delta f\|_{L^q(G)} + \|(\nu, f)\|_{W^{1-\frac{1}{q},q}(\partial G)} + \sum_{i=1}^{B_r} |Z_i|\right),$$

where $c_i = c_i(n,r,q,G)$, $i = 3,4$, are positive constants. As usual, d will denote the exterior derivative and δ the coderivative. Furthermore, we write $\nu \wedge f$ for the tangential component of f and (ν, f) for its normal component. The quantities Y_i and Z_i are real numbers due to the topology of the set G. As usual, B_{n-r} denotes the Betti number regarding $(n-r)$-cycles and B_r means the Betti number concerning r-cycles (see e.g. [13]). In the case of Hölder norms these estimates are generalizations of [2], where vector fields in $\mathbb{R}^3$ are treated. At first sight, the results of [10], [12] and [1] seem to be similar to ours. But there are essential differences that one should well be aware of. Most important, these inequalities are based on orthogonal Hilbert space decompositions, whereas our estimates use Banach norms. To get them in a less general case, G. Schwarz shows in [12] that it is possible to use the ellipticity of a boundary value problem and (if proved) the a priori estimates of this problem. For our proof we choose a direct approach. Regarding the Hilbert space results for vector fields, we discern in [1] an estimation using the fluxes. However this inequality differs from our first one for vector fields and does not concern the $W^{1,2}$-norm. In connection with numerical applications, in [5] several estimates are discussed, which are comparable to ours for 1-forms. However, only Hilbert space results are mentioned there, and even in this case the results are less strict.

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2 Definitions and requirements

If not stated otherwise, let us henceforth assume that $n > 2$, $r \in \mathbb{N}_0 : r \leq n$, $k \in \mathbb{N}_0$, $\lambda \in (0,1)$, $q \in [1,\infty)$ and $\sigma \in \mathbb{R}_+ \cup \{0\}$. Furthermore, we write $G \subset \mathbb{R}^n$ for a bounded set consisting of a finite number of arcwise connected components G_i fulfilling the conditions

$$\overline{G_i} \cap \overline{G_j} = \emptyset \text{ if } i \neq j \text{ and } \partial G_i \text{ belongs to } C^\infty .$$

By $\hat{G}$ the set $\mathbb{R}^n \backslash \overline{G}$ will be denoted. B_r is the abbreviation of the Betti number concerning the simplicial homology group with regard to r-cycles of the set G. We write $f \in X^r$ if f is a r-Form f with euclidian components in the space X. In this discussion it is assumed that the metric is euclidian. The quantities c_i are positive constants relating to our differential forms.

DEFINITIONS:

a) If f and h are r-forms on $\mathbb{R}^n$, then $(f,h)_r$ denotes the euclidian inner product. Let g be a r-form and let $\wedge$ be Graßmann's wedge-product. By means of

$$(*f, g)_{n-r} = (f \wedge g, d^n x)_n$$

we define the Hodge operator $*$ (see e.g. [7]).

b) Let $f \in C^1(G)^r$, $r < n$. Then the exterior derivative d of this form is defined by

$$df := \sum_{1 \leq j_1 < \ldots < j_r \leq n} d(f_{j_1 \ldots j_r}) \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_r} .$$

If f belongs to $C^1(G)^n$, we set $df := 0$.

The codifferential δ is given by means of the Hodge operator $*$ and the exterior differential:

$$\delta f := (-1)^{n(r+1)} * d(*f) .$$

c) Let f be a r-form defined on ∂G and ν be the 1-form with components which are equal to those of the exterior normal field with unit length. Then, by virtue of

$$(\nu f)_{j_1,\ldots,j_{r-1}} := (\nu, f)_{j_1,\ldots,j_{r-1}} := \sum_{i=1}^{n} \nu_i f_{ij_1,\ldots,j_{r-1}} \quad \text{if } r > 0 ,$$

we obtain a differential form νf. The remaining case $r = 0$ will be explained by

$$\nu f := (\nu, f) := 0 .$$

Using the $\wedge$-product, the differential form τf will be designated:

$$\tau f := \nu \wedge f .$$

We call νf normal component and τf tangential component of the form f.

d) As usual, ω_n denotes the surface of the unit ball in $\mathbb{R}^n$ and $G(x,x')$ the fundamental solution of the Laplace operator, i. e.

$$G(x,x') := -\frac{1}{(n-2)\omega_n|x-x'|^{n-2}}, \quad \text{where } x, x' \in \mathbb{R}^n \text{ and } x \neq x'.$$

By the exterior derivative with respect to x, i. e. d_x, the 1-form

$$\Gamma(x,x') := d_x\, G(x,x')$$

is given.

e) If $g(x,.)$ is a integrable r-form, the integral of $g(x,.)$ will be defined as that differential form, the components of which correspond with the integrals of the components of $g(x,.)$.

f) Let $\sigma \in \mathbb{R}_+$. Besides the common spaces, we define

$$\begin{aligned}
&C^{k,\lambda}_\tau(\partial G)^r := \{f \in C^{k,\lambda}(\partial G)^r \mid \tau f = 0\}\\
&W^{\sigma,q}_\tau(\partial G)^r := \{f \in W^{\sigma,q}(\partial G)^r \mid \tau f = 0\}\\
&W^{0,q}_\tau(\partial G)^r := L^q_\tau(\partial G)^r := \{f \in L^q(\partial G)^r \mid \tau f = 0\}\\
&C^{k,\lambda}_\nu(\partial G)^r := \{f \in C^{k,\lambda}(\partial G)^r \mid \nu f = 0\}\\
&W^{\sigma,q}_\nu(\partial G)^r := \{f \in W^{\sigma,q}(\partial G)^r \mid \nu f = 0\}\\
&W^{0,q}_\nu(\partial G)^r := L^q_\nu(\partial G)^r := \{f \in L^q(\partial G)^r \mid \nu f = 0\}\\
&\mathcal{Y}(G)^r := \{f \in C^0(\bar{G})^r \cap C^1(G)^r \mid df = 0 \text{ and } \delta f = 0 \text{ with } \tau f = 0\}\\
&\mathcal{Z}(G)^r := \{f \in C^0(\bar{G})^r \cap C^1(G)^r \mid df = 0 \text{ and } \delta f = 0 \text{ with } \nu f = 0\}.
\end{aligned}$$

Apart from the fact that elements f of the spaces $\mathcal{Y}(\hat{G})^r$ and $\mathcal{Z}(\hat{G})^r$ have to fulfil the property

$$|f_{i_1\ldots i_r}(x)| \to 0 \quad \text{if } |x| \to \infty \text{ uniformly for all directions } [1],$$

these spaces are explained analogous to $\mathcal{Y}(G)^r$ and $\mathcal{Z}(G)^r$ respectively. We call elements of the spaces $\mathcal{Y}(G)^r$ and $\mathcal{Y}(\hat{G})^r$ Dirichlet fields and elements of $\mathcal{Z}(G)^r$ and $\mathcal{Z}(\hat{G})^r$ Neumann fields.

Obviously, we obtain

$$C^k_\nu(\partial G)^0 = C^k(\partial G)^0,\ C^k_\tau(\partial G)^n = C^k(\partial G)^n \text{ and } C^k_\nu(\partial G)^n = C^k_\tau(\partial G)^0 = \{0\}.$$

[1] *Remark:* We take from [9] that this convergence implies that $|f_{i_1\ldots i_r}(x)| = \mathcal{O}(|x|^{1-n})$ if $|x| \to \infty$.

g) Provided that f is a integrable r-form on ∂G, we set

$$Y_i := -\int_{\partial G} (f, \tau \hat{z}^i)\, d\omega\,, \quad i = 1, ..., B_{n-r}\,,$$

and

$$Z_i := -\int_{\partial G} (f, \nu \hat{y}^i)\, d\omega\,, \quad i = 1, ..., B_r\,.$$

Here $(.,.)$ denotes the euclidian inner product. Furthermore, $\{\hat{z}^i\}_{i=1,...,B_{n-r}}$ is a basis of $\mathcal{Z}(\hat{G})^{r-1}$ and $\{\hat{y}^i\}_{i=1,...,B_r}$ is a basis of $\mathcal{Y}(\hat{G})^{r+1}$.

3 A priori estimates

Referring to [9], we define the Dirichlet problem $\mathcal{D}_{\lambda,r}(G)$ and its dual Neumann problem $\mathcal{N}_{\lambda,r}(G)$ for differential forms. It is important to see that suitable integrability conditions be fulfilled. In our case, we use those of [9]. Our a priori inequalities are adequate for these boundary value problems and involve optimal regularity properties.

DEFINITION *(The Dirichlet and Neumann problem): Let $r \in \mathbb{N} : r < n$ and $\lambda \in (0,1)$.*

a) We suppose that $f \in C^{1,\lambda}(\bar{G})^{r+1}$, $g \in C^{1,\lambda}(\bar{G})^{r-1}$ and $\gamma \in C^{1,\lambda}_{\tau}(\partial G)^{r+1}$ satisfy the integrability conditions. By $\{\hat{z}^i\}_{i=1,...,B_{n-r}}$ a basis of $\mathcal{Z}(\hat{G})^{r-1}$ will be designated. The quantities Y_i, $i = 1, ..., B_{n-r}$, are real numbers.
Then $\mathcal{D}_{\lambda,r}(G) = \mathcal{D}_{\lambda,r}(G, f, g, \gamma, \{Y_i \,|\, i = 1, ..., B_{n-r}\})$ denotes the Dirichlet problem, which consists in finding the solution $\phi \in C^{0,\lambda}(\bar{G})^r \cap C^{1,\lambda}(G)^r$ of

$$d\phi = f \quad \text{and} \quad \delta\phi = g \quad \text{in } G$$
$$-\tau\phi = \gamma \quad \text{on } \partial G\,,$$

satisfying the additional conditions

$$-\int_{\partial G} (\phi, \tau \hat{z}^i)\, d\omega = Y_i\,, \quad i = 1, ..., B_{n-r}\,.$$

b) Let us now assume that $g^* \in C^{1,\lambda}(\bar{G})^{r+1}$, $f^* \in C^{1,\lambda}(\bar{G})^{r-1}$ and $\gamma^* \in C^{1,\lambda}_{\nu}(\partial G)^{r-1}$ fulfil the integrability conditions. By $\{\hat{y}^i\}_{i=1,...,B_r}$ a basis of $\mathcal{Y}(\hat{G})^{r+1}$ will be denoted. The quantities Z_i, $i = 1, ..., B_r$, are real numbers.
Then $\mathcal{N}_{\lambda,r}(G) = \mathcal{N}_{\lambda,r}(G, f^*, g^*, \gamma^*, \{Z_i \,|\, i = 1, ..., B_r\})$ means the Neumann problem, which consists in finding the solution $\phi^* \in C^{0,\lambda}(\bar{G})^r \cap C^{1,\lambda}(G)^r$ of

$$d\phi^* = g^* \quad \text{and} \quad \delta\phi^* = f^* \quad \text{in } G$$
$$-\nu\phi^* = \gamma^* \quad \text{on } \partial G\,,$$

satisfying the additional conditions

$$- \int_{\partial G} (\phi^*, \nu \hat{y}^i)\, d\omega = Z_i\,, \quad i = 1, ..., B_r\,.$$

These problems and the Dirichlet and Neumann problems from [8] and [14] are closely related, which can be shown by [7], (3.13, p. 37), (3.10 g, p. 34) and some well-known formulas. Furthermore, each Neumann problem $\mathcal{N}_{\lambda,n-r}(G)$ is equivalent to a Dirichlet problem $\mathcal{D}_{\lambda,r}(G)$. For this, we transform $\mathcal{N}_{\lambda,n-r}(G) = \mathcal{N}_{\lambda,n-r}(G, f^*, g^*, \gamma^*, \{Z_i \mid i = 1, ..., B_{n-r}\})$, by means of

$$\begin{aligned} & g := -(-1)^{n(r+1)} *^{-1} g^*\,, \quad f := (-1)^{nr} *^{-1} f^* \\ & \gamma := (-1)^{nr} *^{-1} \gamma^* \\ & \hat{y}^i := *^{-1} \hat{z}^i\,, \quad Y_i := (-1)^{n(r+1)} Z_i\,, \end{aligned}$$

into $\mathcal{D}_{\lambda,r}(G) = \mathcal{D}_{\lambda,r}(G, f, g, \gamma, \{Y_i \mid i = 1, ..., B_{n-r}\})$. Here, we write $*$ for the Hodge operator and $*^{-1}$ for its inverse mapping. By virtue of the definition $\phi^* := *\phi$, the solution ϕ of $\mathcal{D}_{\lambda,r}(G)$ gives the solution ϕ^* of $\mathcal{N}_{\lambda,n-r}(G)$.

The problem $\mathcal{D}_{\lambda,r}(G)$ and the accompanying problem $\mathcal{N}_{\lambda,r}(G)$ are solved using the Fredholm integral operators $I - K_r$ and $I - L_r$. We will present the compact operators K_r and L_r in Theorem 3.2. The following theorem provides some regularity results for several boundary integral operators. For a detailed proof the reader will be referred to [3], [6] and [15].

THEOREM 3.1: *Let $k \in \mathbb{N}_0$, $\lambda \in (0,1)$, $q \in (1,\infty)$, $\sigma \in \mathbb{R}^+ \cup \{0\}$ and $l, m \in \{1, 2, ..., n\}$. Given $x, x' \in \partial G$, the functions*

$$\begin{aligned} G(x,x') &:= -\frac{1}{(n-2)\omega_n} \cdot \frac{1}{|x-x'|^{n-2}} \\ K(x,x') &:= (\nu(x), \Gamma(x,x')) = (\nu(x), d_x\, G(x,x')) \\ L(x,x') &:= (\nu(x'), \Gamma'(x,x')) = (\nu(x'), d_{x'}\, G(x,x')) \\ N(x,x') = N^{lm}(x,x') &:= (\nu_l(x) - \nu_l(x')) \cdot \Gamma_m(x,x') \end{aligned}$$

will be abbreviated by $O(x,x')$. Moreover, by

$$O_{k,\lambda} f(x) := \int_{\partial G} O(x,x') f(x')\, d\omega_{x'}\,, \quad \text{with } f \in C^{k,\lambda}(\partial G)\,,$$

integral operators $O_{k,\lambda}$ are defined. Analogously, operators $O_{\sigma,q}$ are defined on $W^{\sigma,q}(\partial G)$. Then the operator $O_{k,\lambda}$ belongs to $\mathcal{L}(C^{k,\lambda}(\partial G), C^{k+1,\lambda}(\partial G))$ and $O_{\sigma,q}$ is in $\mathcal{L}(W^{\sigma,q}(\partial G), W^{\sigma+1,q}(\partial G))$.

By means of these regularity results, it is possible to find convenient regularity properties for those integral operators which are related to our boundary problems:

<u>THEOREM 3.2:</u> *Let $r \in \mathbb{N}_0 : r \le n$, $k \in \mathbb{N}_0$, $\lambda \in (0,1)$, $q \in (1,\infty)$ and $\sigma \in \mathbb{R}^+ \cup \{0\}$. We define operators K_r and L_r by*

$$K_r f(x) := K_{k,\lambda,r} f(x) := -2 \int_{\partial G} (\nu(x), d_x(f(x') \cdot G(x,x')))\, d\omega_{x'}\,,$$
$$\text{with } f \in C_\nu^{k,\lambda}(\partial G)^r\,,\ x \in \partial G\,,$$

and

$$L_r f(x) := L_{k,\lambda,r} f(x) := -2 \int_{\partial G} (\nu(x) \wedge \delta_x(f(x') \cdot G(x,x')))\, d\omega_{x'}\,,$$
$$\text{with } f \in C_\tau^{k,\lambda}(\partial G)^r\,,\ x \in \partial G\,.$$

Therein d_x and δ_x denote (with regard to x) the exterior differential and the codifferential respectively. Analogously, we get operators $K_{\sigma,q,r}$ and $L_{\sigma,q,r}$ defined on $W_\nu^{\sigma,q}$ and $W_\tau^{\sigma,q}$ respectively.

Then the operator $K_{k,\lambda,r}$ belongs to $\mathcal{L}(C_\nu^{k,\lambda}(\partial G)^r, C_\nu^{k+1,\lambda}(\partial G)^r)$ and $L_{k,\lambda,r}$ is in $\mathcal{L}(C_\tau^{k,\lambda}(\partial G)^r, C_\tau^{k+1,\lambda}(\partial G)^r)$. Furthermore, $K_{\sigma,q,r}$ lies in $\mathcal{L}(W_\nu^{\sigma,q}(\partial G)^r, W_\nu^{\sigma+1,q}(\partial G)^r)$ and $L_{\sigma,q,r}$ is in $\mathcal{L}(W_\tau^{\sigma,q}(\partial G)^r, W_\tau^{\sigma+1,q}(\partial G)^r)$.

Proof: The assertions for $K_{k,\lambda,n}$ and $L_{k,\lambda,0}$ are trivial. Therefore, we confine ourselves to $K_{k,\lambda,r}$, $r < n$, and $L_{k,\lambda,r}$, $r > 0$. Self-evidently, regarding the Hodge operator and a simple calculation, the first operator is related to the latter. Therefore, it is sufficent to deal merely with $K_{k,\lambda,r}$. Let henceforth $r < n$ and $f \in C_\nu^{0,\lambda}(\partial G)^r$. For $u \in C^1(\mathbb{R}^n)^0$ and $g \in C^1(\mathbb{R}^n)^r$ the equation

$$(3.1) \qquad d(ug) = du \wedge g + u \cdot dg$$

holds. This and the definition of Γ (see *Definitions d)*, part 2) result in

$$(3.2) \qquad d_x(f(x') \cdot G(x,x')) = \Gamma(x,x') \wedge f(x')\,.$$

Futhermore, we make use of the property $(\nu, f)(x') = 0$ and of the relation

$$(3.3) \qquad (\nu(x), \Gamma(x,x') \wedge f(x')) = f(x') \cdot (\nu(x), \Gamma(x,x')) \; - \; \Gamma(x,x') \wedge (\nu(x) - \nu(x'), f(x'))\,.$$

The required estimates for the operators $K_{k,\lambda,r}$ and $L_{k,\lambda,r}$ can now be inferred from those for the operators $K_{k,\lambda}$ and $N_{k,\lambda}$ of *Theorem 3.1.*

The space $C_\nu^{k,\lambda}(\partial G)^r$ lies densely in $W_\nu^{\sigma,q}(\partial G)^r$. Therefore, we verify the results for $K_{\sigma,q,r}$ and $L_{\sigma,q,r}$ by suitable extensions of operators $K_{k,\lambda,r}$, with $k + \lambda > \sigma$, and $L_{k,\lambda,r}$ respectively.

□

Remark: Let $r < n$. By $\langle .,. \rangle \to \mathbb{R}$,

$$(3.4) \qquad \langle f, g \rangle := \int_{\partial G} (f(x), (\nu(x), g(x)))\, d\omega_x\,, \quad \text{with } f \in L^2_\nu(\partial G)^r,\ g \in L^2_\tau(\partial G)^{r+1}\,,$$

a bilinear form is given. The operator $L_{\sigma=0,q=2,r+1}$ is the adjoint operator of $-K_{\sigma=0,q=2,r}$ relating to the bilinear form above (see [9]).

Now, we turn to our main topic: the a priori estimates for $C^{1,\lambda}$-forms concerning $C^{1,\lambda}$- and $W^{1,q}$-norms.

<u>THEOREM 3.3:</u> *Let $r \in \mathbb{N} : r < n$, $\lambda \in (0,1)$, $q \in (1,\infty)$ and $f \in C^{1,\lambda}(\bar{G})^r$.*

There exist positive constants $c_i = c_i(n, r, \lambda, G)$, $i = 1, 2$, such that

$$\|f\|_{C^{1,\lambda}(\bar{G})} \le c_1 \left(\|df\|_{C^{0,\lambda}(\bar{G})} + \|\delta f\|_{C^{0,\lambda}(\bar{G})} + \|\nu \wedge f\|_{C^{1,\lambda}(\partial G)} + \sum_{i=1}^{B_{n-r}} |Y_i|\right)$$

and

$$\|f\|_{C^{1,\lambda}(\bar{G})} \le c_2 \left(\|df\|_{C^{0,\lambda}(\bar{G})} + \|\delta f\|_{C^{0,\lambda}(\bar{G})} + \|(\nu, f)\|_{C^{1,\lambda}(\partial G)} + \sum_{i=1}^{B_r} |Z_i|\right).$$

Moreover, there exist positive constants $c_i = c_i(n, r, q, G)$, $i = 3, 4$, such that

$$\|f\|_{W^{1,q}(G)} \le c_3 \left(\|df\|_{L^q(G)} + \|\delta f\|_{L^q(G)} + \|\nu \wedge f\|_{W^{1-\frac{1}{q},q}(\partial G)} + \sum_{i=1}^{B_{n-r}} |Y_i|\right)$$

and

$$\|f\|_{W^{1,q}(G)} \le c_4 \left(\|df\|_{L^q(G)} + \|\delta f\|_{L^q(G)} + \|(\nu, f)\|_{W^{1-\frac{1}{q},q}(\partial G)} + \sum_{i=1}^{B_r} |Z_i|\right).$$

Proof: As stated by [9], we may decompose a form $f \in C^{1,\lambda}(\bar{G})^r$ into the exterior differential of a $(r-1)$-form ϕ and the codifferential of a $(r+1)$-form θ:

$$(3.5\text{ a}) \qquad f(x) = d\phi(x) + \delta\theta(x) \qquad \text{for } x \in G\,.$$

In this relation, the differential form ϕ is given by

$$(3.5\text{ b}) \qquad \phi(x) := \int_G (\delta f)(x') \cdot G(x, x')\, dx' - \int_{\partial G} (\nu f)(x') \cdot G(x, x')\, d\omega_{x'}$$

and the differential form θ by

$$(3.5\text{ c}) \qquad \theta(x) := \int_G (df)(x') \cdot G(x, x')\, dx' - \int_{\partial G} (\nu \wedge f)(x') \cdot G(x, x')\, d\omega_{x'}\,.$$

Potential theory treats integral operators G_V and G_E given by

$$G_V f(x) := \int_G G(x,x')f(x')\,dx' \quad \text{and} \quad G_E f(x) := \int_{\partial G} G(x,x')f(x')\,d\omega_{x'}\,.$$

Considering the properties of G and ∂G, we take from [3] and [11] that the accompanying operator G_V belongs to

$$\mathcal{L}(C^{0,\lambda}(\bar G), C^{2,\lambda}(\bar G)) \quad \text{or} \quad \mathcal{L}(L^q(G), W^{2,q}(G))\,.$$

Furthermore, it turns out (cf. [3] and [17]) that the operator G_E is in

$$\mathcal{L}(C^{1,\lambda}(\partial G), C^{2,\lambda}(\bar G)) \quad \text{or} \quad \mathcal{L}(W^{1-\frac{1}{q},q}(\partial G), W^{2,q}(G))\,.$$

According to the decomposition (3.5 a-c), we therefore get the $C^{1,\lambda}$-estimate

$$\text{(3.6 a)} \qquad \|f\|_{C^{1,\lambda}(\bar G)} \le c_1\,(\|df\|_{C^{0,\lambda}(\bar G)} + \|\delta f\|_{C^{0,\lambda}(\bar G)} + \|f\|_{C^{1,\lambda}(\partial G)})$$

and the $W^{1,q}$-estimate

$$\text{(3.6 b)} \qquad \|f\|_{W^{1,q}(G)} \le c_2\,(\|df\|_{L^q(G)} + \|\delta f\|_{L^q(G)} + \|f\|_{W^{1-\frac{1}{q},q}(\partial G)})\,.$$

The form $\epsilon(x) := -(\nu, f)(x)$, describing the normal component of f, satisfies the inhomogenous integral equation

$$\text{(3.7 a)} \qquad (I - K_{r-1})\epsilon = \mu\,,$$

where μ is defined by

$$\text{(3.7 b)} \qquad \mu(x) := \mu(df, \delta f, \nu\wedge f)(x) := -2\,(\nu(x), d\int_G ((\delta f)(x')\cdot G(x,x'))\,dx' + \\ +\delta[\int_G ((df)(x')\cdot G(x,x'))\,dx' - \int_{\partial G}((\nu\wedge f)(x')\cdot G(x,x'))\,d\omega_{x'}]) \\ \text{for } x\in\partial G\,.$$

For this differential form the estimates

$$\text{(3.8 a)} \qquad \|\mu\|_{C^{1,\lambda}(\partial G)} \le c_3\,(\|\delta f\|_{C^{0,\lambda}(\bar G)} + \|df\|_{C^{0,\lambda}(\bar G)} + \|\nu\wedge f\|_{C^{1,\lambda}(\partial G)})$$

and

$$\text{(3.8 b)} \qquad \|\mu\|_{W^{1-\frac{1}{q},q}(\partial G)} \le c_4\,(\|\delta f\|_{L^q(G)} + \|df\|_{L^q(G)} + \|\nu\wedge f\|_{W^{1-\frac{1}{q},q}(\partial G)})$$

hold. Now, we write $\langle .,. \rangle$ for the bilinear form defined in (3.4). According to [3] and [9] the Riesz index of K_r equals to 1. Consequently, to each basis $\{e_i\}_{i=1,\dots,B_{n-r}}$ of $\mathcal{N}(I+L_r)$ there exists a basis $\{e_i^*\}_{i=1,\dots,B_{n-r}}$ of $\mathcal{N}(I-K_{r-1})$ satisfying

$$\text{(3.9 a)} \qquad \langle e_i^*, e_j\rangle = \int_{\partial G}(e_i^*(x), (\nu(x), e_j(x)))\,d\omega_x = \delta_{ij}\,, \quad i,j\in\{1,\dots,B_{n-r}\}\,.$$

For details of the latter conclusion see [14], (Satz I.2.4, especially Hilfssatz I.3.8). We take from [9] that $\{(\nu, e_i)\}_{i=1,\dots,B_{n-r}}$ is a basis of $\mathcal{Z}(\hat G)^{r-1}|_{\partial G}$ and that it is possible to choose

$e_i = \nu \wedge \hat{z}^i$, where $\hat{z}^i \in \mathcal{Z}(\hat{G})^{r-1}$ and $i = 1, ..., B_{n-r}$. That is why the orthogonality relation (3.9 a) gives

$$(3.9\text{ b})\qquad \int_{\partial G} (e_i^*(x), \hat{z}^j(x))\, d\omega_x = \delta_{ij}\,, \quad i, j \in \{1, ..., B_{n-r}\}\,.$$

Assuming h belongs to $\mathcal{R}(I - K_{r-1})$, we presuppose that g is a solution of

$$(3.10)\qquad (I - K_{r-1})g = h\,, \quad \text{where} \quad \int_{\partial G} (g, \hat{z}^i)\, d\omega = Y_i\,, \quad i = 1, ..., B_{n-r}\,.$$

By the regularity properties of K_{r-1} one verifies that g belongs to $C_\nu^{1,\lambda}(\partial G)^{r-1}$. In addition, we define

$$(3.11)\qquad g_1 := g - g_0 \quad \text{and} \quad g_0 = g_0(Y_1, ..., Y_{B_{n-r}}) := \sum_{i=1}^{B_{n-r}} Y_i e_i^*\,.$$

Hence, one infers

$$(3.12)\qquad (I - K_{r-1})g_1 = h \quad \text{and} \int_{\partial G} (g_1, \hat{z}^i)\, d\omega = 0\,, \quad i = 1, ..., B_{n-r}\,.$$

As mentioned above, the Riesz index of K_r is equal to 1. Consequently, the property $h \notin \mathcal{N}(I - K_{r-1})$ implies that $g_1 \in C_\nu^{1,\lambda}(\partial G)^{r-1} \backslash \mathcal{N}(I - K_{r-1})$ [2]. Furthermore, the embeddings

$$J_1 : C_\nu^{2,\lambda}(\partial G)^{r-1} \longrightarrow C_\nu^{1,\lambda}(\partial G)^{r-1} \quad \text{and}$$
$$J_2 : W_\nu^{2-\frac{1}{q},q}(\partial G)^{r-1} \longrightarrow W_\nu^{1-\frac{1}{q},q}(\partial G)^{r-1}$$

are compact. Instead of $J_1 \circ K_{1,\lambda,r-1}$, we merely write $K_{1,\lambda,r-1}$, and we substitute $K_{1-\frac{1}{q},q,r-1}$ for $J_2 \circ K_{1-\frac{1}{q},q,r-1}$. Hence,

$$I - K_{1,\lambda,r-1} \in \mathcal{L}(C_\nu^{1,\lambda}(\partial G)^{r-1}, C_\nu^{1,\lambda}(\partial G)^{r-1}) \quad \text{and}$$
$$I - K_{1-\frac{1}{q},q,r-1} \in \mathcal{L}(W_\nu^{1-\frac{1}{q},q}(\partial G)^{r-1}, W_\nu^{1-\frac{1}{q},q}(\partial G)^{r-1})$$

are Fredholm operators and have closed ranges. Therefore, Banach's *open mapping theorem*

[2] *Remark:* The fact that g_1 belongs to

$$(\{f \in C_\nu^{1,\lambda}(\partial G)^{r-1} \mid \int_{\partial G} (f, \hat{z}^i)\, d\omega = 0, i = 1, ..., B_{n-r}\}, \|.\|_{C^{1,\lambda}(\partial G)})$$

or to

$$(\{f \in W_\nu^{1-\frac{1}{q},q}(\partial G)^{r-1} \mid \int_{\partial G} (f, \hat{z}^i)\, d\omega = 0, i = 1, ..., B_{n-r}\}, \|.\|_{W^{1-\frac{1}{q},q}(\partial G)})$$

respectively provides another possibility to obtain the estimates (3.13 a,b).

yields

(3.13 a) $\quad \|g_1\|_{C^{1,\lambda}(\partial G)} \le c_5 \|(I - K_{1,\lambda,r-1})g_1\|_{C^{1,\lambda}(\partial G)}$

and

(3.13 b) $\quad \|g_1\|_{W^{1-\frac{1}{q},q}(\partial G)} \le c_6 \|(I - K_{1-\frac{1}{q},q,r-1})g_1\|_{W^{1-\frac{1}{q},q}(\partial G)}\,.$

We derive from (3.7 a), (3.11), (3.13 a,b) and [9] (the proof of Satz 8.3) that

(3.14 a) $\quad \|(\nu, f)\|_{C^{1,\lambda}(\partial G)} \le c_7(\|\mu\|_{C^{1,\lambda}(\partial G)} + \sum_{i=1}^{B_{n-r}} |Y_i|)$

and

(3.14 b) $\quad \|(\nu, f)\|_{W^{1-\frac{1}{q},q}(\partial G)} \le c_8(\|\mu\|_{W^{1-\frac{1}{q},q}(\partial G)} + \sum_{i=1}^{B_{n-r}} |Y_i|)\,.$

Then $f_{|\partial G}$ will be decomposed into its tangential and normal part. Finally, the estimates (3.6 a,b), (3.8 a,b) and (3.14 a,b) will be combined. The corresponding dual results follow quite analogously.

□

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